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# Field models from two-cocycles on infinite-dimensional Lie groups and symplectic structures: 2D-gravity and Chern–Simons theory

Victor Aldaya†‡, José Navarro-Salas‡§ and Miguel Navarro†‡

† Departamento de Física Teórica y del Cosmos, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain

‡ IFIC, Centro Mixto Universidad de Valencia-CSIC, Burjassot 46100-Valencia, Spain

§ Departamento de Física Teórica, Facultad de Física, Universidad de Valencia, Burjassot 46100-Valencia, Spain

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**Abstract.** In this paper the connection between field models and infinite-dimensional Lie groups is widely analysed on the bases of a new group quantization approach. We also relate the Poincaré–Cartan form of variational calculus to the symplectic current/structure of the covariant phase-space formulation of (higher-derivative) field theory. The Virasoro and Kac–Moody groups are considered. In the first case the action functional of the 2D-induced gravity in the light-cone formulation is derived. The hidden  $SL(2, \mathbb{R})$  simply appears as generated by the kernel of the Lie algebra two-cocycle and plays the role of a gauge-type symmetry. Nevertheless, it is shown that a proper space-like formulation is out of reach of the Virasoro group. The corresponding symplectic structure of the (non-local) action functional is determined showing that it is related to the symplectic structure associated with the  $SL(2, \mathbb{R})$ -Kac–Moody group. This unravels the proper geometrical meaning of the hidden symmetry and differs from the analysis in related works based on the coadjoint-orbit approach. The relation between the Kac–Moody groups and the Chern–Simons gauge theory on a disc in the presence of a source is considered using the new approach.

## 1. Introduction

In this paper we give a systematic analysis of the relationship between field models and infinite-dimensional Lie groups. Our point of view is based on a previously introduced group theoretical quantization approach [1–3] that generalizes in several aspects the geometric quantization and the coadjoint-orbit method [4–6]. At the classical level the method [1–3] aims to construct an action functional from a Lie group wearing a  $U(1)$ -fibred structure (as do, for instance, centrally extended groups like the Virasoro and Kac–Moody groups). The simplest example for illustrating this method is that of  $\tilde{\mathbb{G}}_{(m)}$  the central extension of the Galilei group  $\mathbb{G}$  by  $U(1)$ , where the parameter  $m$  (the mass) is an element of  $H^2(\mathbb{G}, U(1))$ . Choosing the following 2-cocycle for fulfilling the central extension:

$$\xi_{(m)}(g', g) = m[x' \cdot v + t(v \cdot v + \frac{1}{2} V'^2)]$$

the Maurer–Cartan left-invariant 1-form associated with the central generator is

$$\Theta = -m\mathbf{x} \cdot d\mathbf{v} - \frac{1}{2}mv^2 dt + \frac{d\zeta}{i\zeta}$$

where  $\zeta = \exp i\varphi$  is the  $U(1)$  central parameter.  $\Theta$  is also related to the Poincaré–Cartan form of classical mechanics (see later). When restricted to trajectories  $\Theta$  becomes

$$(m\mathbf{v} \cdot \dot{\mathbf{x}} - \frac{1}{2}mv^2) dt = \mathcal{L} dt$$

up to a total derivative. In this case the group  $\tilde{\mathbb{G}}_{(m)}$  is a trivial  $U(1)$ -bundle on  $\tilde{\mathbb{G}}_{(m)}/U(1)$ . In general, as we shall see later in this paper, the non-triviality of the  $U(1)$ -fibred structure leads to multivalued action functionals. The peculiarities in the quantization procedure will be discussed in the next sections.

As a by-product of our approach we provide a natural construction of the so called ‘hidden symmetries’. Since we construct Lagrangians from Lie groups as associated with a particular left-invariant form  $\Theta$ , the right-invariant vector fields on the group leave the Lagrangian automatically invariant generating then the ordinary phase space symmetries. Furthermore, the Lie algebra of the group is also realized by means of left-invariant vector fields. Nevertheless, only those left-generators in the kernel of the two-cocycle (the kernel of  $d\Theta$  as well) leave the Lagrangian invariant and generate a sort of ‘hidden’ gauge-type symmetry.

Extending the results of [7], we consider the Virasoro group and obtain the action functional of the 2D-induced gravity model [8] in the same way as the previous example, and in agreement with previous works [9, 10]. The hidden  $SL(2, \mathbb{R})$  symmetry is generated by the left-invariant vector fields in the kernel of the Lie algebra two-cocycle. However, a more careful analysis shows that this connection only appears in the light-cone formulation, i.e. when  $x^+$  plays the role of the evolution coordinate, and the standard space-like formulation requires new ingredients for a proper definition of the symplectic structure of the Polyakov model [8]. This important point has not been considered in related works based on the coadjoint-orbit approach [9, 10].

We relate the Poincaré–Cartan form of the higher-order variational calculus to the symplectic current of the covariant phase space formulation of field theory. Our proposal for the symplectic current for higher-derivative theories can also be used to define the symplectic structure of non-local field theories. We can then provide the proper expression of the symplectic structure, in the space-like formulation, of the 2D-induced gravity despite of the non-local character of the Lagrangian. In the space-like formulation, the 2D-induced gravity is no longer attached to the starting Lie group (Virasoro) structure. In fact, it turns out to be properly associated with the (non-reduced)  $SL(2, \mathbb{R})$ -Kac–Moody group clarifying then the proper geometrical meaning of the hidden symmetries of the theory.

The physical picture that arises from our analysis is the following. In the light-cone formulation, the 2D-induced gravity is geometrically associated with the Virasoro group. The hidden  $SL(2, \mathbb{R})$  symmetry emerges then as a ‘left’ symmetry (i.e. a gauge-type symmetry generated by the left-invariant vector fields in the kernel of the Lie algebra two-cocycle). In the space-like formulation, the theory is geometrically attached to the (non-reduced)  $SL(2, \mathbb{R})$ -Kac–Moody group and the hidden symmetry is then a *right* symmetry (i.e. phase space symmetry generated by the right-invariant vector fields on the group).

The application of our approach to the Kac–Moody group  $LG$  leads directly to the action functional of the Chern–Simons (cs) gauge theory on the disc in the space-like

formulation. Obviously we avoid the two-steps interpretation of the corresponding field model: first as a WZW-model in the light-cone formulation [9] and then their explicit equivalence with the CS theory on the disc [11]. We also show how we obtain an action functional describing the CS theory in the presence of a source by simply adding some cohomologically trivial two-cocycles to the Kac-Moody group.

We carry out the quantization of the CS theory on the disc from the new quantization method and obtain a Hilbert space given by the integrable Kac-Moody representations, in agreement with other methods [12]. The quantization of the field model defined by the Virasoro group, i.e. the 2D-induced gravity model in the light-cone formulation, is much more complicated. Classically, their phase spaces coincide with the coadjoint orbits,  $\text{diff } S^1/S^1$  and  $\text{diff } S^1/SL(2, \mathbb{R})$  of the Virasoro group. The standard geometric quantization of the coadjoint-orbits fails because it only gives rise to non-degenerated Verma module representations [13]. In terms of the present quantization scheme it is possible to solve this problem by imposing 'higher-order polarization' conditions to the quantum wavefunctions (see [14] for details).

The paper is organized as follows. In section 2, and based on elaborated aspects of the variational calculus and the Noether theorem, we provide a generalization of the 'symplectic current' for higher-order and non-local Lagrangians to be used later. Section 3 follows with a short summary of the quantization approach we use in this paper. In section 4 we consider the field model constructed on the Virasoro group, the Virasoro field model, in connection with the 2D-induced gravity. We also work out the symplectic structure of the (non-local) Polyakov action in the space-like formulation showing the proper meaning of the hidden  $SL(2, \mathbb{R})$ -Kac-Moody symmetry. In section 5 we consider the field models constructed from the Kac-Moody groups in connection with the Chern-Simons gauge theory. By introducing a family of cohomologically 'trivial' two-cocycles for the Kac-Moody group we describe the theory in the presence of a source. The quantization of the model is then carried out on the bases of the new group quantization approach.

## 2. The variational calculus and the symplectic current for higher-order derivatives

The main achievement of this section is the formulation of the Noether theorem for Lagrangians depending on a field  $\psi^\alpha(x)$  and its derivatives  $\partial^s \psi^\alpha / \partial x^{\mu_1} \dots \partial x^{\mu_s}$ ,  $s = 1, \dots, r$ , in such a way that we can find a generalization of the 'symplectic current' and, hence, of the symplectic form, for arbitrary Lagrangians.

First of all we shall become familiar with the fundamentals of the higher-order variational calculus in the most economical way, restricting ourselves to the ordinary Hamilton principle. For a deeper study of this subject we refer the reader to [15] (see also [16]).

According to the ordinary Hamilton Principle, the higher-order variational calculus starts with a real Lagrangian density  $L = L(\psi^\alpha, \partial_\mu \psi^\alpha, \dots, \partial_{\mu_1 \dots \mu_r} \psi^\alpha)$  and only the variations of the field  $\psi^\alpha$ ,  $\delta \psi^\alpha$ , are considered as being independent. The variations of  $\partial_\mu \psi^\alpha$ , and higher derivatives are induced by those of  $\psi^\alpha$  and  $x^\mu$ ,  $\delta x^\mu$ . This means that

$$\delta \partial_{\mu_1 \dots \mu_s} \psi^\alpha = \frac{d^s}{dx^{\mu_1} \dots dx^{\mu_s}} \delta \psi^\alpha \tag{2.1}$$

where  $df/dx^\mu = \partial f / \partial x^\mu + \partial f / \partial \psi^\alpha \cdot \partial_\mu \psi^\alpha + \partial f / \partial \psi^\alpha_{\nu} \cdot \partial_{\mu\nu} \psi^\alpha + \dots$ , for any function  $f = f(x^\mu, \psi^\alpha, \partial_\mu \psi^\alpha \dots)$ .

The variations of  $x^\mu$ ,  $\psi^\alpha$ ,  $\partial_\mu \psi^\alpha \equiv \psi^\alpha_{,\mu}$ ,  $\partial_{\mu\nu} \psi^\alpha \equiv \psi^\alpha_{,\mu\nu}$ , ... are codified very often by means of a vector field  $X^r = X^\mu \partial / \partial x^\mu + X^\alpha \partial / \partial \psi^\alpha + X^\alpha_\mu \partial / \partial \psi^\alpha_\mu + \dots + X^\alpha_{\mu_1 \dots \mu_r} \partial / \partial \psi^\alpha_{\mu_1 \dots \mu_r}$ , where  $X^\mu$  stands for  $\delta x^\mu$ ,  $X^\alpha$  for  $\delta \psi^\alpha$ ,  $X^\alpha_\mu$  for  $\delta \psi^\alpha_{,\mu}$ , etc. For the ordinary Hamilton principle  $X^r$  is of the form

$$\bar{X}^r = X^\mu \frac{\partial}{\partial x^\mu} + X^\alpha \frac{\partial}{\partial \psi^\alpha} + \bar{X}^\alpha_\mu \frac{\partial}{\partial \psi^\alpha_\mu} + \dots + \bar{X}^\alpha_{\mu_1 \dots \mu_r} \frac{\partial}{\partial \psi^\alpha_{\mu_1 \dots \mu_r}} \quad (2.2)$$

with  $\bar{X}^\alpha_{\mu_1 \dots \mu_s} = d\bar{X}^\alpha_{\mu_1 \dots \mu_{s-1}} / dx^{\mu_s}$ .

The action functional  $I$  is defined as follows:

$$I(\psi) = \int_M \mathcal{L}(\psi, \partial_\mu \psi, \dots, \partial_{\mu_1 \dots \mu_r} \psi) \omega \quad (2.3)$$

where  $\omega$  is the volume on the space-time  $M$ . The Hamilton principle states that the action must be extremal, i.e.

$$\delta_\psi I(X) = \int_M L_{\bar{X}^r}[\mathcal{L}(\psi, \partial_\mu \psi, \dots, \partial_{\mu_1 \dots \mu_r} \psi) \omega] \quad (2.4)$$

for any vector field  $X = X^\mu \partial / \partial x^\mu + X^\alpha \partial / \partial \psi^\alpha$ .

The solutions to (2.4) satisfy the obvious generalization of the first-order Lagrangian equations:

$$\sum_{s=0}^r (-)^s \frac{d^s}{dx^{\mu_1} \dots dx^{\mu_s}} \frac{\partial \mathcal{L}}{\partial \Psi^\alpha_{\mu_1 \dots \mu_s}} = 0. \quad (2.5)$$

### 2.1. The Noether theorem

We first review the case  $r=1$  very briefly. A vector field  $Y = Y^\mu \partial / \partial x^\mu + Y^\alpha \partial / \partial \psi^\alpha$  is said to be a symmetry of the variational problem if

$$L_{\bar{Y}^1}(\mathcal{L}\omega) = d\Delta \quad (2.6)$$

where  $\bar{Y}^1$  is defined as in (2.2) for  $r=1$  and  $\Delta$  is a  $(n-1)$ -form ( $n = \dim M$ )  $\Delta^\mu \theta_\mu$ ,  $\theta_\mu = (i\partial / \partial x^\mu) \omega$ , the components of which,  $\Delta^\mu$ , only depend on  $x^\nu$  and  $\psi^\alpha$ .

Let us assume, for the sake of simplicity, that the variation  $\delta x^\mu$  associated with  $Y$ , i.e. its component  $Y^\mu$ , is zero (under the integral in (2.3) such a transformation can be absorbed by means of a change of variable), and that  $\Delta = 0$ . In this case (2.6) acquires the traditional form

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi^\alpha} \delta \psi^\alpha + \frac{\partial \mathcal{L}}{\partial \psi^\alpha_\mu} \delta \psi^\alpha_\mu \quad \delta \psi^\alpha \equiv Y^\alpha \quad \delta \psi^\alpha_\mu \equiv \bar{Y}^\alpha_\mu. \quad (2.7)$$

Restricting  $\delta \mathcal{L}$  to an arbitrary solution  $\psi$  we have

$$\begin{aligned} \delta \mathcal{L}|_{\text{sol}} &= \left( \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} \right) \delta \psi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} \delta \partial_\mu \psi^\alpha|_{\text{sol}} = \left( \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} \right) \delta \psi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} \frac{d}{dx^\mu} \delta \psi^\alpha|_{\text{sol}} \\ &= \frac{d}{dx^\mu} \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} \delta \psi^\alpha \right) \Big|_{\text{sol}} \equiv \frac{d}{dx^\mu} j^\mu \Big|_{\text{sol}} = 0 \quad j^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\alpha} Y^\alpha. \end{aligned} \quad (2.8)$$

If  $\Delta \neq 0$  then the conserved Noether current must be modified in the form  $j^\mu =$

$\partial\mathcal{L}/\partial\partial_\mu\psi^\alpha \cdot Y^\alpha - \Delta^\mu$ . In the same way, if  $Y^\mu \neq 0$  the current takes the final form

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^\alpha} Y^\alpha + \mathcal{L}Y^\mu - \Delta^\mu. \tag{2.9}$$

The expression (2.9) for the current can be brought to a more compact form with the introduction of the Poincaré–Cartan  $n$ -form  $\Theta_{\text{PC}}$ , which reduces to  $\mathcal{L}\omega$  when  $\psi_\mu^\alpha = \partial_\mu\psi^\alpha$  as has been indeed assumed hitherto. Defining  $\Theta_{\text{PC}}$  as

$$\Theta_{\text{PC}} = \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} (d\psi^\alpha - \psi_\nu^\alpha dx^\nu) \wedge \theta_\mu + \mathcal{L}\omega \equiv (d\psi^\alpha - \psi_\nu^\alpha dx^\nu) \wedge \Omega_\alpha + \mathcal{L}\omega \tag{2.10}$$

the current above takes the form

$$j = *(i_{\mathcal{Y}}\Theta_{\text{PC}} - \Delta) \tag{2.11}$$

where the  $*$  is the duality Hodge operator turning  $(n-1)$ -forms into 1-forms.

Before giving the expression for  $j$  for arbitrary  $r$  let us repeat steps (2.7) and (2.8) for  $r=2$ . In this case

$$\begin{aligned} \mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi^\alpha} \delta\psi^\alpha + \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} \delta\psi_\mu^\alpha + \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \delta\psi_{\mu\nu}^\alpha & \delta\psi^\alpha &\equiv Y^\alpha \\ \delta\psi_\mu^\alpha &\equiv \bar{Y}_\mu^\alpha & \delta\psi_{\mu\nu}^\alpha &\equiv \bar{Y}_{\mu\nu}^\alpha \end{aligned} \tag{2.12}$$

and restricting again  $\delta\mathcal{L}$  to an arbitrary solution  $\psi$  we obtain

$$\begin{aligned} \delta\mathcal{L}|_{\text{sol}} &= \left( \frac{d}{dx^\mu} \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} - \frac{d}{dx^\mu} \frac{d}{dx^\nu} \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \right) \delta\psi^\alpha + \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} \frac{d}{dx^\mu} \delta\psi^\alpha + \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \frac{d}{dx^\mu} \frac{d}{dx^\nu} \delta\psi^\alpha \\ &\quad + \frac{d}{dx^\mu} \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \frac{d}{dx^\nu} \delta\psi^\alpha|_{\text{sol}} \\ &= \frac{d}{dx^\mu} \left[ \left( \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} - \frac{d}{dx^\nu} \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \right) Y^\alpha \right] + \frac{d}{dx^\nu} \left[ \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \bar{Y}_\nu^\alpha \right] \Big|_{\text{sol}} \\ &= \frac{d}{dx^\mu} j^\mu \Big|_{\text{sol}} = 0 & j^\mu &\equiv \left( \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} - \frac{d}{dx^\nu} \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \right) Y^\alpha + \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \bar{Y}_\nu^\alpha. \end{aligned} \tag{2.13}$$

An expression similar to (2.11) can be written for  $j$  in order 2 if we define  $\Theta_{\text{PC}}$  in a form which generalizes (2.10) non-trivially. We write

$$\begin{aligned} \Theta_{\text{PC}} &= \left( \frac{\partial\mathcal{L}}{\partial\psi_\mu^\alpha} - \frac{d}{dx^\nu} \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} \right) (d\psi^\alpha - \psi_\sigma^\alpha dx^\sigma) \wedge \theta_\mu + \frac{\partial\mathcal{L}}{\partial\psi_{\mu\nu}^\alpha} (d\psi_\nu^\alpha - \psi_{\nu\sigma}^\alpha dx^\sigma) \wedge \theta_\mu + \mathcal{L}\omega \\ &\equiv (d\psi^\alpha - \psi_\nu^\alpha dx^\nu) \wedge \Omega_\alpha + (d\psi_\nu^\alpha - \psi_{\nu\sigma}^\alpha dx^\sigma) \wedge \Omega_\alpha^\nu + \mathcal{L}\omega. \end{aligned} \tag{2.14}$$

We now provide general formulas for arbitrary  $r$ . If the vector field  $Y$  is a symmetry of  $\mathcal{L}$ , i.e.

$$L_{\mathcal{Y}}(\mathcal{L}\omega) = d\Delta \tag{2.15}$$

the current

$$j = *(i_{\mathcal{Y}}\Theta_{\text{PC}} - \Delta) \tag{2.16}$$

satisfies

$$\partial_\mu j^\mu|_{\text{sol}} = 0$$

where  $\Theta_{PC}$  is given by [15]:

$$\Theta_{PC} = \sum_{s=1} \gamma_{\mu_1 \dots \mu_{s-1}}^\alpha \wedge \Omega_\alpha^{\mu_1 \dots \mu_{s-1}} + \mathcal{L}\omega$$

$$\Omega_\alpha^{\mu_1 \dots \mu_{s-1}} \equiv \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\mu_1 \dots \mu_{s-1}}^\alpha} + \lambda_{\alpha}^{\mu_1 \dots \mu_{s-1} \mu} \right) \theta_\mu \tag{2.17}$$

where the  $\lambda$ s are solution of the following equations for arbitrary section field  $\Psi^r = (\psi^\alpha, \psi_\mu^\alpha, \psi_{\mu\nu}^\alpha, \dots, \psi_{\mu_1 \dots \mu_r}^\alpha)$ :

$$\lambda_{\alpha}^{\mu_1 \dots \mu_s} |_{\Psi^r} \omega = d\Omega_{\alpha}^{\mu_1 \dots \mu_s} |_{\Psi^r} \quad s = 1, \dots, r-1$$

$$\lambda_{\alpha}^{\mu_1 \dots \mu_r} |_{\Psi^r} = 0.$$

*2.2. The symplectic current*

Going back to the first-order Lagrangians we can introduce the covariant momenta  $\pi_\alpha^\mu \equiv \partial \mathcal{L} / \partial \Psi_\mu^\alpha$  and the covariant Hamiltonian  $\mathcal{H} \equiv \pi_\alpha^\mu \Psi_\mu^\alpha - \mathcal{L}$ . The Poincaré-Cartan form then becomes

$$\Theta_{PC}^\dagger = \pi_\alpha^\mu d\Psi^\alpha \wedge \theta_\mu + \mathcal{H}\omega. \tag{2.18}$$

As we shall see the Poincaré-Cartan form is specially suitable for the definition of the symplectic structure on the phase space of a field theory. The expression (2.18) suggests the definition of a vector-valued one-form on the solution manifold

$$j^\mu = \pi_\alpha^\mu \delta \Psi^\alpha \tag{2.19}$$

where  $\delta$  refers to the exterior derivative on the space of classical solutions. The differential of  $j^\mu$

$$\omega^\mu = -\delta j^\mu = \delta \Psi^\alpha \wedge \delta (\partial \mathcal{L} / \partial \Psi_\mu^\alpha) \tag{2.20}$$

has been called ‘symplectic current’ [17] since it defines a covariant charge—the symplectic form—when integrated on an initial-value hypersurface  $\Sigma$ :

$$\omega = \int_\Sigma d\sigma_\mu \omega^\mu. \tag{2.21}$$

We have to remark that the two-form (2.21) is indeed a presymplectic form. In general,  $\omega$  can be degenerated and this means that we face a constrained system. The true symplectic form and the reduced phase space are then obtained by taking the quotient by the vector fields in the kernel of  $\omega$ . These vector fields are generators of gauge transformations.

In order to be able to provide a natural generalization of these expressions for more general Lagrangians, say higher-order or non-local ones, it is useful to notice that  $j^\mu$  in (2.19) can be defined as the current such that

$$\delta \mathcal{L} = \partial_\mu j^\mu \tag{2.22}$$

on the space of classical solutions of the equation of motion [18].

Defining, as before, the symplectic current as

$$\omega^\mu = -\delta j^\mu \tag{2.23}$$

$\omega^\mu$  verifies:

$$(a) \delta \omega^\mu = 0 \quad (b) \partial_\mu \omega^\mu = 0. \tag{2.24}$$

The last equality, which allow us to interpret  $\omega^\mu$  as a sort of conserved Noether current for  $\Delta = 0$  and  $\delta x^\mu = 0$ , follows since

$$\partial_\mu \omega^\mu = -\delta(\partial_\mu j^\mu) = -\delta(\delta \mathcal{L}) = 0. \tag{2.25}$$

From the corresponding expression for the Poincaré–Cartan form we can write the general expression for the symplectic current potential  $j^\mu$  in the case of a higher-order Lagrangian:

$$j^\mu = \sum_{s=1}^r \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\mu_1 \dots \mu_{s-1} \mu}^\alpha} + \lambda_{\alpha}^{\mu_1 \dots \mu_{s-1} \mu} \right) \delta \Psi_{\mu_1 \dots \mu_{s-1}} \tag{2.26a}$$

where

$$\lambda_{\alpha}^{\mu_1 \dots \mu_{s-1} \mu} = -\frac{d}{dx^\mu} \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\mu_1 \dots \mu_{s-1} \mu}^\alpha} + \lambda_{\alpha}^{\mu_1 \dots \mu_{s-1} \mu} \right). \tag{2.26b}$$

For first-order Lagrangians we recover the expression (2.20) and for the second-order ones we obtain

$$\omega^\mu = \delta \Psi^\alpha \wedge \delta \left( \frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} \right) \right) + \delta \Psi_\nu^\alpha \wedge \delta \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} \right). \tag{2.27}$$

It is obvious that no general expression for the ‘symplectic current’ for non-local Lagrangians can be given but it is clear the way in which it could be obtained.

### 3. A group-theoretic, dynamical method: The Virasoro and Kac–Moody groups

The group-theoretic method we use in this paper [1–3] aims to construct both the classical and quantum dynamics of a physical system characterized by a Lie group  $\tilde{G}$  with a preferred  $U(1)$  subgroup generated by  $\Xi$ . This subgroup defines which group parameter have a coordinate-momentum character and which play, on the contrary, a role similar to that of time. Parameters of  $q$ - $p$  type are those whose corresponding Lie algebra commutator gave a  $\Xi$  term on the RHS. The rest are considered as generalized evolution parameters.  $\tilde{G}$  generalizes the  $U(1)$ -principal bundle structure of a quantum manifold in the geometric quantization (GQ) approach [4–6], now allowing for non-symplectic variables like time. As in GQ we have a (connection) one-form  $\Theta$ , the quantization form, although here it is defined naturally as the component of the left-invariant one-form dual to  $\Xi$  and is not a contact form, i.e.  $d\Theta$  has a kernel and is therefore a presymplectic form. The module  $\ker \Theta \cap \ker d\Theta$  is generated by left-invariant vector fields generating in turn a subalgebra  $\mathcal{G}_\Theta$ , the characteristic subalgebra.

The characteristic subalgebra plays a significant role in many respects.  $\mathcal{G}_\Theta$  constitutes the set of generalized equation of motion. The trajectories of its vector fields reproduce the classical equation of motion as well as the Bohr–Sommerfeld quantization rules, which appear as integrability condition for the (new)  $U(1)$  components of those equations. Moreover, the notion of polarization in this scheme generalizes the GQ analogue in that it contains the subalgebra  $\mathcal{G}_\Theta$ . A polarization is thus a left subalgebra of  $\tilde{G}$  containing  $\mathcal{G}_\Theta$  and excluding  $\Xi$ . The exclusion of the  $\Xi$  generator guarantee that no pair of coordinate-momentum variables are in the polarization  $\mathcal{P}$  contains, roughly speaking,  $\mathcal{G}_\Theta$  as well as half the coordinates and momenta. The polarization conditions  $X^L \Psi = 0$ ,  $X^L \in \mathcal{P}$ , on wavefunctions  $\Psi: \tilde{G} \rightarrow \mathbb{C}$  such that  $\Xi \Psi = i\Psi$ , are required to render the quantum representation irreducible. The Schrödinger equation is one of

them. The quantum operators are the right-invariant vector fields which act properly on the subspace of polarized wavefunctions, as a consequence of the null commutator  $[X^L, X^R] = 0$ , valid for any Lie group. We shall not insist any longer on the quantization techniques, which can be found in [1-3] (see also [14]), but rather we shall be concerned with symmetry properties.

All physical properties of the system characterized by  $\tilde{G}$  are concentrated in the one-form  $\Theta$  generalizing the Poincaré-Cartan form  $\Theta_{PC}$ . In local coordinates  $\Theta$  is of the form  $\Theta = p dq - H dt + d\zeta/i\zeta$ ,  $\zeta \in U(1)$ . The total differential term  $d\zeta/i\zeta$  that distinguishes  $\Theta$  from  $\Theta_{PC}$  converts semi-invariance of the latter into true invariance (null Lie derivative) of the former and permits taking  $\Theta$  to the quotient manifold  $\tilde{G}/\mathcal{G}_\Theta$ . The symmetries of  $\Theta$  are given by right-invariant vector fields  $\tilde{X}^R$ . For them  $L_{\tilde{X}^R}\Theta = 0$  since  $\Theta$  is a component of the left-invariant one-form  $\vartheta^L$ ,  $\Theta \equiv \vartheta^{L(\zeta)}$ . In fact, it is a general property of Lie groups that  $L_{X^R}\vartheta^L = 0$ ,  $L_{X^L(a)}\vartheta^{L(b)} = -C_{ab}^c\vartheta^{L(c)}$  where  $C_{ab}^c$  are the structure constants. The Noether theorem is then nothing but the assertion that  $L_{\tilde{X}^L}(i_{\tilde{X}^R}\Theta) = 0$ ,  $\forall \tilde{X}^L \in \mathcal{G}_\Theta$ , i.e.  $i_{\tilde{X}^R}\Theta$ —the Noether invariants—are constants along the generalized trajectories of the motion.

In addition to the right-invariant vector fields (RIVF), those left-invariant vector fields (LIVF) in  $\mathcal{G}_\Theta$  also leave  $\Theta$  invariant and constitute, therefore, a (non-conventional) symmetry of the physical system. In fact,  $L_{\tilde{X}^L(a)}\vartheta^{L(\zeta)} = -C_{ab}^c\vartheta^{L(b)}$  and no vector field  $\tilde{X}^L_{(a)}$  there exists whose commutator with  $\tilde{X}^R_{(a)} \in \mathcal{G}_\Theta$  gives a term proportional to  $\tilde{X}^L_{(a)} \equiv \Xi$ . None the less  $i_{\tilde{X}^L(a)}\vartheta^{L(\zeta)} = 0$  by duality, i.e. the left Noether invariants are zero. The relevance of the left-symmetry will come out later when translated to ordinary field theory.

### 3.1. The Virasoro group

Let us now analyse the case where  $\tilde{G}$  is the extended diffeomorphism group of  $S^1$ ,  $\text{diff } S^1$ . Given a diffeomorphism  $F: S^1 \mapsto S^1$ , we can define co-ordinates  $l_n$ ,  $n \in \mathbb{Z}$ , by

$$F(\sigma) = \sigma + \sum_{n \in \mathbb{Z}} l_n e^{in\sigma} \quad \sigma \in S^1 \quad (3.1)$$

so that the group law  $(F * G)(\sigma) = G \circ F(\sigma) \equiv G(F(\sigma))$  adopts the expression ( $l'$  corresponds to  $F$  and  $l''$  to  $F * G$ ):

$$l''_m = l'_m + l_m + ip l'_p l'_{m-p} + \frac{(ip)^2}{2!} l'_p l'_n l'_{m-n-p} + \dots \quad (3.2)$$

The two-cocycle giving the central extension of  $\text{diff } S^1$  by  $U(1)$  reads, when written in coordinates  $l_n$  [14],

$$\xi(F, G) = -\frac{c}{24} \sum_{r,k=1} \sum_{m_1+\dots+m_r+n_1+\dots+n_k=0} \frac{(-i)^{k+r-1}}{k!r!} (r-1)! m_1 \dots m_r (n_1 + \dots + n_k) \times P^{(k)}(n_1, \dots, n_k) l_{n_1} \dots l_{m_r} l'_{n_1} \dots l'_{n_k} \quad (3.3)$$

where  $P^{(k)}(n_1, \dots, n_k)$  is a symmetric, homogeneous polynomial of degree  $k$  in the  $n_1, \dots, n_k$  variables. Its expression is

$$P^{(k)}(n_1, \dots, n_k) = \sum_{\text{part}(k)} a_{\lambda_1 \dots \lambda_s}^{(k)} n_1^{\lambda_1} \dots n_s^{\lambda_s} \quad (3.4)$$

where the coefficients  $a_{\lambda_1 \dots \lambda_s}^{(k)}$  are obtained recursively by

$$a_{\lambda_1 \dots \lambda_s}^{(k)} = \begin{cases} (k-1)! \text{ if } [\lambda_1, \dots, \lambda_s] = [1, \dots, 1] \\ \sum_{i=1}^k a_{\lambda_1 \dots \lambda_i - 1 \dots \lambda_s}^{(k-1)}. \end{cases} \tag{3.5}$$

In local coordinates any  $g \in \text{diff } S^1$  can be finally given as  $g = (G, \zeta)$ ,  $G \in \text{diff } S^1$ ,  $\zeta \in U(1)$ . Then the composition law  $g'' = g' * g$  is written as follows:

$$g'' = (G \circ F, \zeta' \zeta \exp i\xi(F, G) \exp i\xi_{\text{cob}}(F, G)) \tag{3.6}$$

where we have also added a trivial cocycle (coboundary [14] generated by the function  $l_0$ , i.e.

$$\xi_{\text{cob}} = -\frac{c'}{24} (l_0'' - l_0' - l_0).$$

Here  $c'$  is a constant parameter and  $l_0''$  is given by (3.2).

From (3.6) all the physical information of this model can be drawn out. We want now to rewrite  $\Theta$ ,  $\tilde{X}^{L_s}$  and  $\tilde{X}^{R_s}$  in 'coordinates'  $F$  to compare with more physical objects. To do that let us start with  $\Theta$  in coordinates  $l_n$  and perform the change of variables (3.1) *a posteriori*.  $\Theta$  is given by:

$$\Theta = \frac{-i}{24} (k^3 c - kc') l_{-k} dl_k - \frac{c}{24} k^2 \sum_{j=2}^{\infty} \sum_{n_1 + \dots + n_j = -k} (-i)^j n_1 \dots n_j l_{n_1} \dots l_{n_j} dl_k + \frac{d\zeta}{i\zeta}. \tag{3.7}$$

Making the mentioned change  $\Theta$  becomes

$$\Theta = \frac{c}{48\pi} \int_0^{2\pi} d\sigma \left( \frac{dF''}{F'} \right) - \frac{c'}{48\pi} \int_0^{2\pi} d\sigma ((F' - 1) dF) + \frac{d\zeta}{i\zeta} \tag{3.8}$$

and after two integration by parts,

$$\Theta = -\frac{c}{48\pi} \int d\sigma \frac{dF}{F'} \left( \frac{F'''}{F'} - 2 \frac{F''^2}{F'^2} \right) - \frac{c'}{48\pi} \int d\sigma dF (F' - 1) + \frac{d\zeta}{i\zeta}. \tag{3.9}$$

The  $F$ -expression of the vector fields is nevertheless more easily obtained by first working out the unextended part from the group law of  $\text{diff } S^1$ , and then the central terms by duality on (3.9). The expressions of  $\tilde{X}_{F(\sigma)}^R$  and  $\tilde{X}_{F(\sigma)}^L$  are respectively

$$\tilde{X}_{F(\sigma)}^R = F'(\sigma) \frac{\delta}{\delta F(\sigma)} - \frac{1}{48\pi} \left\{ c \left( \frac{F'''}{F'} - \frac{F''^2}{F'^2} \right) + c'(F' - 1) \right\} (\sigma) \Xi \tag{3.10a}$$

$$\tilde{X}_{F(\sigma)}^L = \int d\varepsilon \delta(F(\varepsilon) - \sigma) \left[ \frac{\delta}{\delta F(\varepsilon)} + \frac{1}{48\pi} \left\{ c'(F' - 1) + c \frac{1}{F'} \left( \frac{F'''}{F'} - 2 \frac{F''^2}{F'^2} \right) \right\} (\varepsilon) \Xi \right]. \tag{3.10b}$$

As mentioned above the ordinary symmetry is realized by the right-invariant vector fields, and the group-theoretic version of the Noether theorem provides the Noether invariants [7]

$$\begin{aligned} \left( \tilde{X}_{l_n}^{L,R} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{inF(\sigma)} \tilde{X}_{F(\sigma)}^{L,R} \right) \\ i_{\tilde{X}_{l_k}^R} \Theta = -\frac{1}{24\pi} \int_0^{2\pi} d\sigma e^{ik\sigma} \left[ c \left( \frac{F'''}{F'} - \frac{3}{2} \frac{F''^2}{F'^2} \right) + \frac{c'}{2} F'^2 \right] \equiv \int_0^{2\pi} d\sigma e^{ik\sigma} T \end{aligned} \tag{3.11}$$

i.e. the Fourier coefficients of an energy-momentum tensor.

It can be checked that for  $\tilde{X}^L$  in the characteristic subalgebra  $\mathcal{G}_\Theta$ , i.e.  $\langle \tilde{X}_0^L, \tilde{X}_1^L \rangle$  if  $c'/c = r^2 \in \mathbb{N}^2$ , or  $\langle \tilde{X}_0^L \rangle$  otherwise,  $L_{\tilde{X}^L} \Theta = 0$  but the Noether charge  $i_{\tilde{X}^L} \Theta$  is actually zero.

### 3.2. Kac-Moody groups

In the Kac-Moody case we start with the (symmetry) group  $\tilde{G} \equiv LG$ , the central extension by  $U(1)$  of the loop group,  $LG$ , on a connected, simply connected, simple, compact group  $G$ . As is well known,  $LG$  is a non-trivial principal bundle with base  $G$  and fibre  $U(1)$  [12, 19]. The Maurer-Cartan one-form  $\Theta$  provides a globally defined connection one-form in the principal bundle  $LG \mapsto G$ . However, due to the non-triviality of the bundle we cannot find a well-defined projection of  $\Theta$  onto the physical 'configuration' space  $G$  and, hence, a well-defined, univalued Lagrangian. Nevertheless, using a local trivialization of the bundle, we can provide a local expression for  $\Theta$  and then a (now multivalued) action functional.

Given two elements  $(a, \zeta)$ ,  $(b, \zeta')$  of a local trivialization of  $LG$  such that  $a, b$  and  $a \cdot b$  are in a neighbourhood of the identity of  $G$ , the composition law has the general form

$$(b, \zeta') * (a, \zeta) = (b \cdot a, \zeta' \zeta \exp i\xi(b, a)) \quad (3.12)$$

where the 'two-cocycle'  $\xi$  is given by [19]

$$\begin{aligned} \xi(b, a) &= 2\pi k \left\{ \frac{-1}{16\pi^2} \int_D \langle b^{-1} \cdot db, da \cdot a^{-1} \rangle \right\} \\ &+ \int_D \{H(Z) - H(X) - H(Y)\} \\ H(X) &\equiv \langle dX, h(ad X) dX \rangle \\ h(Z) &\equiv -\frac{1}{48\pi^2} [(\sinh Z - Z)/Z^2] \end{aligned} \quad (3.13)$$

and  $b \cdot a \equiv \exp Z$ ,  $a \equiv \exp X$ ,  $b \equiv \exp Y$  (the dot between elements of  $G$  will henceforth be omitted). The bracket  $\langle , \rangle$  in (3.13) is the Killing form of the Lie algebra of  $G$  with the standard normalization and  $D \subset \mathbb{R}^2$  is the unit disc ( $\partial D = S^1$ ). Of course, the expression (3.12) is well-defined irrespective of the extension of  $g$  from  $S^1$  to  $D$  [20]. For the sake of simplicity we shall use the same notation for maps on  $S^1$  and for any extension of them to  $D$ . The two-form  $H$  verifies the property

$$dH(X) = \frac{1}{48\pi^2} \langle da a^{-1}, (da a^{-1})^2 \rangle. \quad (3.14)$$

The 'winding number'  $k \in \mathbb{Z}$  will play the role of the dimensionless coupling constant of the physical action (the Chern-Simons gauge theory [11], as we shall show later).

The cocycle (3.13) is given as an integral of a two-form on  $D$ , i.e.

$$\xi(b, a) = \int_D (b, a)^* \alpha_2 \quad (3.15)$$

$\alpha_2$  being a two-form on  $G \times G$  and  $(a, b)^*$  the pull back of the cartesian map  $a \times b \equiv (a, b)$  (an extension to be precise) from  $D$  to  $G \times G$ . From  $\xi(a, b)$  the local expression of the left-invariant one-form  $\Theta \equiv \vartheta^{L(\xi)}$  can be immediately derived according to the

general formula

$$\Theta = \frac{\partial \xi(g', g)}{\partial g} \Big|_{g'=g^{-1}} dg + \frac{d\xi}{i\xi}. \tag{3.16}$$

To carry on the local calculation of  $\vartheta^{L(\xi)} \equiv \Theta$  we shall write (3.15) as the integral of a one-form  $\alpha_1$ :

$$\xi(b, a) = \int_{S^1} (b, a)^* \alpha_1. \tag{3.17}$$

However, we need not the actual expression of  $\alpha_1$ . It suffices to write  $\alpha_1$  as

$$\alpha_1 = A_i(b, a) da^i + B_i(b, a) db^i. \tag{3.18}$$

Then, the one-form  $\vartheta^{L(\xi)}$  acquires the general form

$$\vartheta^{L(\xi)} = \frac{d\xi}{i\xi} - \int_{S^1} d\sigma \left\{ \left[ \frac{\partial A_i}{\partial a^j} - \frac{\partial A_j}{\partial a^i} \right] a^{ij} da^i + \left[ \frac{\partial B_i}{\partial b^j} - \frac{\partial B_j}{\partial b^i} \right] b^{ij} da^i \right\} \Big|_{b=a^{-1}} \tag{3.19}$$

where the prime stands for the derivative with respect to the  $S^1$  argument.

On the other hand  $d\alpha_1 = \alpha_2$  takes the form

$$\begin{aligned} \alpha_2 = d\alpha_1(b, a) &= \frac{1}{2} \left[ \frac{\partial A_i}{\partial a^j} - \frac{\partial A_j}{\partial a^i} \right] da^j \wedge da^i + \left[ \frac{\partial A_i}{\partial b^j} - \frac{\partial B_j}{\partial a^i} \right] db^j \wedge da^i \\ &+ \frac{1}{2} \left[ \frac{\partial B_i}{\partial b^j} - \frac{\partial B_j}{\partial b^i} \right] db^j \wedge db^i. \end{aligned} \tag{3.20}$$

Comparing (3.20) and (3.19), and taking into account the expression of  $\xi(b, a)$  in (3.13) we can identify terms and write a local expression for  $\vartheta^{L(\xi)}$  as

$$\begin{aligned} \vartheta^{L(\xi)} &= \frac{d\xi}{i\xi} + 2\pi k \left\{ \frac{-1}{16\pi^2} \int_{S^1} d\sigma \langle b^{-1} b', da a^{-1} \rangle \Big|_{b=a^{-1}} \right. \\ &+ \left. \int_{S^1} d\sigma [\langle X', h(\text{ad}X) dX \rangle - \langle dX, h(\text{ad}X) X' \rangle] \right\} \\ &= \frac{d\xi}{i\xi} + 2\pi k \left\{ \frac{-1}{16\pi^2} \int_{S^1} d\sigma \langle a^{-1} a', a^{-1} da \rangle \right. \\ &+ \left. \int_{S^1} d\sigma [\langle X', h(\text{ad}X) dX \rangle - \langle dX, h(\text{ad}X) X' \rangle] \right\} \end{aligned} \tag{3.21}$$

As in the Virasoro case and for the sake of completeness, let us consider the ordinary symmetries. Let  $X_a^R$  be the RIVF on  $G$ . The extension,  $\tilde{X}^R$ , of the right-invariant vector fields on  $LG$  could be calculated directly from the group law (3.12) but it is far simpler to the invariance properties of  $\vartheta^L$  with respect to  $\tilde{X}^R$ . In fact,  $L_{\tilde{X}^R} \vartheta^L \equiv (i_{\tilde{X}^R} d + di_{\tilde{X}^R}) \vartheta^L = 0$ . Then if  $\tilde{X}^R$  is of the general form  $X^R + X^{R\zeta} \Xi$ , one has

$$\begin{aligned} 0 &= L_{\tilde{X}^R} \vartheta^{L(\xi)} = i_{X^R} d\vartheta^{L(\xi)} + d(i_{X^R} \vartheta^{L(\xi)} + X^{R\zeta}) = L_{X^R} \vartheta^{L(\xi)} + dX^{R\zeta} \\ &\Rightarrow L_{X^R} \vartheta^{L(\xi)} = -dX^{R\zeta} \end{aligned} \tag{3.22}$$

which determines  $X^{R\zeta}$  completely since  $X^{R\zeta}$  must vanish at the identity of the group.

The extended RIVFs take the form:

$$\begin{aligned} \tilde{X}_{a^i(x)}^R = X_{a^i(x)}^R + 2\pi k \left[ \frac{1}{16\pi^2} \langle X_{a^i}, a^i a^{-1} \rangle \right. \\ \left. - i_{X_{a^i(x)}} [\langle X^i, h(\text{ad}X) dX \rangle - \langle dX, h(\text{ad}X) X^i \rangle] \right] \Xi \end{aligned} \tag{3.23}$$

$$\tilde{X}_\zeta^R = i_\zeta \frac{\partial}{\partial \zeta} \Xi$$

Following the general scheme, the Noether invariants associated with the right-symmetry (3.23) are given by

$$i_{\tilde{X}_{a^i(x)}^R} \Theta = \frac{k}{4\pi} \langle X_{a^i}, a^i a^{-1} \rangle. \tag{3.24}$$

Obviously, the Poisson brackets of the invariants (3.24) close the Lie algebra of the symmetry group  $\overline{LG}$ . The two-dimensional nature of the Kac-Moody field model will be discussed in section 5.

To conclude this section let us consider the variables  $a_n^i$ ,  $n \in \mathbb{Z}$  in analogy to (3.1):

$$a^i(\sigma) = \sum_{n \in \mathbb{Z}} a_n^i e^{in\sigma}. \tag{3.25}$$

The associated generators (say left-) will be given by

$$X_{a_n^i}^{L_i} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{in\sigma} X_{a^i(\sigma)}^{L_i}. \tag{3.26}$$

It is not difficult, in this variables, to work out the characteristic subalgebra (a basis of which generates the module  $\text{Ker } \Theta \cap \text{Ker } d\Theta$ ). It is

$$\mathcal{G}_\Theta = \langle X_{a_0^i}^{L_i}, i = 1, \dots, \dim G \rangle. \tag{3.27}$$

#### 4. 2D-induced gravity

The integral of the 1-form  $\Theta$  for the Virasoro group along the ‘trajectories’  $dF \equiv \dot{F} d\tau$  defines the following action functional:

$$S = \int \Theta = -\frac{c}{48\pi} \int d\sigma d\tau \frac{\dot{F}}{F'} \left( \frac{F'''}{F'} - 2 \frac{F''}{F'^2} \right) - \frac{c'}{48\pi} \int d\sigma d\tau (F' - 1) \dot{F} \tag{4.1}$$

where the  $\tau$  variable is the evolution parameter (the dot means derivative with respect to  $\tau$ ). From now on this field will be referred to as the Virasoro field model.

##### 4.1. The Virasoro field model and hidden (left) symmetries in 2D-gravity

The Virasoro field model whose action is given by (4.1), defines a two-dimensional conformal field model—invariant under the transformations  $x^- \rightarrow h(x^-)$ ,  $x^+ \rightarrow \bar{h}(x^+)$ —if the ‘space’ parameter  $\sigma$  and the evolution coordinate  $\tau$  are interpreted as the light-cone coordinates  $x^-$ ,  $x^+$  (or  $x^+$ ,  $x^-$ ) respectively. With this identification the action (4.1)

now reads

$$S = \int dx^+ dx^- \left\{ \frac{-c}{48\pi} \frac{F_+}{F_-} \left( \frac{F_{--}}{F_-} - 2 \frac{F_{-}^2}{F_-^2} \right) - \frac{c'}{48\pi} F_+(F_- - 1) \right\} \equiv \int \mathcal{L}\omega. \tag{4.2}$$

This action functional coincides, for  $c' = 0$ , with the 2D-induced gravity action in the light-cone gauge [8] (where  $f = F^{-1}$ ; had we chosen the group law for  $\text{diff } S^1$  as  $(F * G)(x) = F(G(x))$  instead of  $(F * G)(x) = G(F(x))$  we would have obtained directly the Polyakov action [8]). An expression similar to (3.9, 4.2) has also been obtained in [9] in a different way by means of the co-adjoint orbit method and parametrizing the orbit with the group variables. It should be noted that the requirement of using the group parameters in constructing the action functional in the co-adjoint orbit method (see also [10]) partially parallels the idea of the group theoretical quantization [1-3] at the classical level (at the quantum level the difference between both approaches is much deeper (see [14])).

We have to point out that the evolution parameter for the Virasoro field model should be identified with a light-cone coordinate in order to relate it to the gravitational action. The Polyakov action (4.2) in the standard space-like formulation is out of reach of the Virasoro group. We shall see later in this section, and after constructing the proper symplectic form of 2D-gravity, that it can be still related with other Lie groups. Before that we can explore the ‘hidden’ symmetries of the Virasoro field model and translate them to 2D-gravity.

As mentioned in section 2, the action (4.1-4.2) has the ordinary (right) symmetry-diff  $S^1$ —generated by the right-invariant vector fields  $\tilde{X}_i^R$  leading to the corresponding conserved charges (3.11). In addition, and following the general scheme, we can face ‘hidden’ (left) symmetries generated by the left-invariant vector fields lying in the characteristic subalgebra  $\mathcal{G}_\Theta$  or, equivalently, in the kernel of the two-cocycle. In fact those vector fields  $X^L$  in  $\mathcal{G}_\Theta$  are characterized by the defining property that  $i_{\tilde{X}^L}\Theta = 0 = i_{\tilde{X}} d\Theta$ , which implies  $L_{\tilde{X}^L}\Theta = 0$ , i.e. they are symmetries in the physical sense although obviously the Noether charges  $i_{\tilde{X}^L}\Theta$  are zero. However, a realization of  $\mathcal{G}_\Theta$  on the space  $(\tau, \sigma; F)$  of definition of the physical action can produce a non-trivial Noether current. It seems then clear that a field-theoretic version of the vector fields in  $\mathcal{G}_\Theta$  can be the generators of the hidden symmetry. In this case the hidden (left-)symmetry depends upon the values of  $c$  and  $c'$ . If  $c'/c = r^2 \in \mathbb{N}^2$  the hidden symmetry is generated by  $\mathcal{G}_\Theta = \langle \tilde{X}_{l_0}^L, \tilde{X}_{l_r}^L, \tilde{X}_{l_r'}^L \rangle \approx sL(2, \mathbb{R})$ , and  $\mathcal{G}_\Theta = \langle \tilde{X}_{l_0}^L \rangle$  for  $c'/c \neq r^2$ .

It must be remarked that, in general, the RIVFs on a Lie group  $\tilde{G}$  pass on the manifold on which it acts from the left, giving a realization of the Lie algebra  $\mathcal{G}$  of  $G$ . In the present case all the RIVFs of the Virasoro group are translated to the  $(\tau, \sigma; F)$  space defining the ordinary (unextended) symmetry of (4.2). Nevertheless, only those LIVFs of the Virasoro group that belong to  $\mathcal{G}_\Theta$  define an action on the  $(\tau, \sigma; F)$  space

$$\begin{aligned} X_{l_0}^L &= \int d\sigma \frac{\delta}{\delta F(\sigma)} \\ X_{l_r}^L &= \int d\sigma e^{\pm rF(\sigma)} \frac{\delta}{\delta F(\sigma)} \end{aligned} \tag{4.3}$$

leaving semi-invariant the Poincaré-Cartan form  $\Theta_{PC}$  associated with the third-order Lagrangian in (4.2), i.e.  $L_{\tilde{X}^L}\Theta_{PC} = d\Delta_{X^L}$ . The appearance of  $\Delta_{X^L}$  is traced back to the missing of the central extension terms in going from the group to the  $(\tau, \sigma; F)$  space.

Following the general scheme of section 2 the form  $\Theta_{\text{PC}}$  is

$$\begin{aligned}\Theta_{\text{PC}} &= \mathcal{L}\omega + \gamma \wedge \left( \frac{\partial \mathcal{L}}{\partial F_\mu} + \lambda^\mu \right) \theta_\mu + \gamma_\nu \wedge \left( \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} + \lambda^{\mu\nu} \right) \theta_\mu + \gamma_{\nu\sigma} \wedge \frac{\partial \mathcal{L}}{\partial F_{\mu\nu\sigma}} \theta_\mu \\ &= \mathcal{L}\omega + \gamma \wedge \frac{\partial \mathcal{L}}{\partial F_+} \theta_+ + \gamma \wedge \left( \frac{\partial \mathcal{L}}{\partial F_-} + \lambda^- \right) \theta_- + \gamma_- \wedge \left( \frac{\partial \mathcal{L}}{\partial F_{--}} + \lambda^{--} \right) \theta_- + \gamma_{--} \\ &\quad \wedge \frac{\partial \mathcal{L}}{\partial F_{---}} \theta_{--}.\end{aligned}$$

An explicit calculation gives

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial F_+} &= \frac{-c}{48\pi} \left\{ \frac{1}{F_-} \left( \frac{F_{---}}{F_-} - 2 \frac{F_{--}^2}{F_-^2} \right) + r^2 (F_- - 1) \right\} \\ \frac{\partial \mathcal{L}}{\partial F_-} + \lambda^- &= \frac{-c}{48\pi} \left\{ \frac{F_{+--}}{F_-^2} + r^2 F_+ \right\} \\ \frac{\partial \mathcal{L}}{\partial F_{--}} + \lambda^{--} &= \frac{-c}{48\pi} \left\{ \frac{-F_{+-}}{F_-^2} - 2 \frac{F_+ F_{--}}{F_-^3} \right\} \\ \frac{\partial \mathcal{L}}{\partial F_{---}} &= \frac{-c}{48\pi} \frac{F_+}{F_-^2}.\end{aligned}\tag{4.5}$$

Let us unify the notation for the vector fields (4.3) as

$$X_k \equiv e^{ikF} \frac{\partial}{\partial F} \quad k=0 \quad \pm r.$$

The expression of their extension  $\bar{X}_k^3$  acting on  $F, F_\mu, F_{\mu\nu\lambda}$ , is given by

$$\begin{aligned}\bar{X}_k^3 &= e^{ikF} \left\{ \frac{\partial}{\partial F} + (ik) F_\mu \frac{\partial}{\partial F_\mu} + [(ik)^2 F_\mu F_\nu + (ik) F_{\mu\nu}] \frac{\partial}{\partial F_{\mu\nu}} \right. \\ &\quad \left. + [(ik)^3 F_\mu F_\nu F_\lambda + (ik)^2 (F_{\mu\nu} F_\lambda + F_{\lambda\mu} F_\nu + F_{\nu\lambda} F_\mu) + (ik) F_{\mu\nu\lambda}] \frac{\partial}{\partial F_{\mu\nu\lambda}} \right\}.\end{aligned}\tag{4.6}$$

To test the semi-invariance of the Poincaré-Cartan form we can restrict ourselves to the case  $F_\mu = \partial_\mu F, F_{\mu\nu} = \partial_\nu F_\mu$ , so that  $L_X \Theta_{\text{PC}} = L_X (\mathcal{L}\omega)$  and, in this case, it reduces to  $L_X \mathcal{L}\omega$ . The Lie derivative of  $\mathcal{L}$  gives

$$\begin{aligned}\bar{X}_k^3 \mathcal{L} &= k^2 \left[ ik F_+ (F_- - 1) + \frac{F_+ F_{--}}{F_-} \right] = -\partial_\mu \Delta_{(k)}^\mu \\ \Delta_{(k)}^\mu &\equiv ik e^{ikF} \left[ \frac{F_{--}}{F_-} + ik (F_- - 1) - \frac{F_{+-}}{F_-} \right].\end{aligned}\tag{4.7}$$

The corresponding conserved currents are

$$J_{(0)} = -\frac{c}{48\pi} \left[ r^2 (F_- - 1) + \left( \frac{F_{---}}{F_-^2} - 2 \frac{F_{--}^2}{F_-^3} \right), r^2 F_+ + \frac{F_{+--}}{F_-^2} \right]\tag{4.8}$$

$$J_{(\pm)} = -\frac{c}{48\pi} e^{\pm irF} \left[ (\pm ir) \frac{F_{--}}{F_-} + \left( \frac{F_{---}}{F_-^2} - 2 \frac{F_{--}^2}{F_-^3} \right), \frac{F_{+--}}{F_-^2} - 2(\pm ir) \frac{F_{+-}}{F_-} - (\pm ir) \frac{F_+ F_{--}}{F_-^2} \right].$$

These currents can be written in the form  $J^\mu \equiv (J^+, J^-) = (\partial_- \tilde{J}^+, J^-)$ , in agreement with the fact that the Noether charges in the light-cone,  $\int dx^- J^+$ , associated with hidden symmetries are zero. Indeed, since  $\Delta$  above is defined except for a closed form,  $J$  is defined up to a divergenceless term  $J'$ . Taking  $J' = (-\partial_- \tilde{J}^+, \partial_+ \tilde{J}^+)$  we get  $J + J' = (0, j)$ , satisfying  $\partial_- j = 0$ , where  $j$  is given by

$$\begin{aligned}
 j_{(0)} &= \frac{-c}{24\pi} \left[ \frac{F_{+--}}{F_-^2} - \frac{F_{--}F_{+-}}{F_-^3} + r^2 F_+ \right] \\
 j_{(\pm)} &= \frac{-c}{24\pi} e^{\pm irF} \left[ \frac{F_{+--}}{F_-^2} - \frac{F_{--}F_{+-}}{F_-^3} - (\pm ir) \frac{F_{+-}}{F_-} \right].
 \end{aligned}
 \tag{4.9}$$

Needless to say that the ordinary Noether charges defined on the plane  $t=0$  are non-trivial ( $\int dx(J^+ + J^-) \neq 0$ ).

If we now consider that  $\sigma \in \mathbb{R}$  instead of  $\sigma \in S^1$ , the expression of the one-form  $\Theta$  is the same as in (3.9) but the characteristic subalgebra  $\mathcal{G}_\Theta$  is a  $sl(2, \mathbb{R})$  algebra (even for  $c' = 0$ ) generated by the vector fields

$$X_{(-)}^L = \int d\sigma \frac{\delta}{\delta F(\sigma)} \quad X_{(0)}^L = \int d\sigma F(\sigma) \frac{\delta}{\delta F(\sigma)} \quad X_{(+)}^L = \int d\sigma F^2(\sigma) \frac{\delta}{\delta F(\sigma)}.
 \tag{4.10}$$

A calculation similar to the one described above leads to the following Noether currents

$$\begin{aligned}
 j_{(-)} &= -\frac{c}{24\pi} \left[ \frac{F_{+--}}{F_-^2} - \frac{F_{--}F_{+-}}{F_-^3} \right] \\
 j_{(0)} &= -\frac{c}{24\pi} \left[ \frac{FF_{+--}}{F_-^2} - \frac{FF_{--}F_{+-}}{F_-^3} - \frac{F_{+-}}{F_-} \right] \\
 j_{(+)} &= -\frac{c}{24\pi} \left[ \frac{F^2 F_{+--}}{F_-^2} - \frac{F^2 F_{--}F_{+-}}{F_-^3} - 2 \frac{FF_{+-}}{F_-} + 2F_+ \right].
 \end{aligned}
 \tag{4.11}$$

These conserved currents coincide with the ‘accidental’  $sl(2, \mathbb{R})$  current algebra of 2D-induced gravity discovered in [8]. In fact, it is straightforward to verify that after the change of variables  $x^- \rightarrow F(x^-)$ , the metric field  $h_{++} \equiv f_+/f_-$  (where  $ds^2 = dx^+ dx^- + h_{++}(dx^+)^2$ ) takes the form

$$f_+/f_- \approx j_{(+)}(x^+) - 2j_{(0)}(x^+) \cdot x^- + j_{(-)}(x^-).
 \tag{4.12}$$

#### 4.2. Symplectic structure of the non-local 2D-gravity action and hidden (right) symmetries

It is clear from the results of this section and section 3 that  $d\Theta$  provides the symplectic form of 2D-gravity for the hypersurface  $x^+ = 0$ . In fact,  $d\Theta$  coincides with one of the chiral components of the symplectic current,  $\omega^+$ , of the action (4.2) integrated with respect to  $x^-$ . To get the symplectic form associated with the hypersurface  $t=0$  we need to know both chiral components  $\omega^+$  and  $\omega^-$  of the symplectic current. We can compute  $\omega^-$  by means of the general expression for the symplectic current obtained in the previous section. Up to total derivatives, the symplectic current is given by (we

have identified the symbols  $d$  and  $\delta$ )

$$\omega^- = -\frac{c}{48\pi} \left[ dF \wedge d\left(\frac{F_{+--}}{F_-^2}\right) - dF_- \wedge d\left(\frac{F_{+-}}{F_-^2} + 2\frac{F_+ F_{--}}{F_-^3}\right) + dF_{--} \wedge d\left(\frac{F_+}{F_-^2}\right) \right] + \frac{c'}{48\pi} dF \wedge dF_+ \quad (4.13a)$$

$$\omega^+ = \frac{c}{48\pi} dF \wedge d\left(\frac{F_{---}}{F_-^2} - 2\frac{F_{--}}{F_-^3}\right) + \frac{c'}{48\pi} dF \wedge dF_- \quad (4.13b)$$

The symplectic form for the hypersurface  $t=0$  is then

$$\omega = \int dx(\omega^- + \omega^+). \quad (4.14)$$

We have to stress that  $\omega$  is not degenerated, thus leading to a well-defined Poisson bracket, even though the chiral symplectic forms  $\int dx^- \omega^+$  and  $\int dx^+ \omega^-$  have non-trivial kernel (this issue and its relation with the Poisson-Lie groups have been discussed in [18]). We can now conclude immediately that the 2D-induced gravity action (4.2), in the space-like frame, is no longer attached to the Virasoro group. In other words,  $d\Theta$  cannot define properly the Poisson bracket for the currents (4.11), due to the fact that they become trivial functions on the phase space defined by the hypersurface  $x^+ = 0$ .

To investigate the possibility of finding a new Lie group capturing, in the space-like formulation, the Polyakov model we have to reconsider the gauge-fixed action functional in terms of the metric field  $h(\equiv h_{++})$  instead of the auxiliary field  $F$  (or  $f$ )

$$S = \frac{c}{24\pi} \int \partial_-^2 h \nabla^{-1} \partial_- h \quad (4.15)$$

where  $\nabla = \partial_+ - h\partial_-$ . Although this action is non-local, the formalism developed in section 2 allow us to carry out a canonical treatment of the theory.

The equation of motion reads

$$2\partial_-^2(\nabla^{-1}\partial_- h) - (\partial_- \nabla^{-1}\partial_- h)^2 = 0 \quad (4.16)$$

and the symplectic current potential is given by

$$j^+ = -(\partial_- h + h\partial_- \nabla^{-1}\partial_- h)\delta(\nabla^{-1}\partial_- h) - 2\partial_- \nabla^{-1}\partial_- h \delta h \quad (4.17a)$$

$$j^- = \partial_- \nabla^{-1}\partial_- h \delta(\nabla^{-1}\partial_- h). \quad (4.17b)$$

The symplectic form for the hypersurface  $t=0$  is then

$$\omega = \frac{c}{24\pi} \int (\omega^- + \omega^+). \quad (4.18)$$

It is not difficult to see that the equation (4.16) can be rewritten as

$$\{f, x^-\} = 0 \quad (4.19)$$

where  $\{f, x^-\}$  is the Schwartzian derivative and  $\nabla^{-1}\partial_- h = \ln \partial_- f$ . As is well known the general solution to (4.19) is

$$f = \frac{A(x^+)x^- + B(x^+)}{C(x^+)x^- + D(x^+)} \quad (4.20)$$

where  $AD - CD = 1$ . It is now easy to arrive at the standard equation [8] ( $h = \partial_+ f / \partial_- f$ ):

$$\partial^3 h = 0. \tag{4.21}$$

Inserting the solution (4.20) into (4.18), and after a long but straightforward calculation, we obtain the following expression for  $\omega$ :

$$\omega = \frac{c}{6\pi} \int dy \left\{ \delta(B'D - BD') \delta(C/D) - \frac{1}{D^2} \delta D \delta D' \right\} \tag{4.22}$$

It is not difficult to check now that the two-form  $\omega$  (4.22) turns out to be related with the corresponding two-form  $d\Theta$  associated with the  $SL(2, \mathbb{R})$ -Kac-Moody group (for  $SL(2, \mathbb{R})$  we can still use the expression (3.21)). The form  $\omega$  is invariant under a constant  $SL(2, \mathbb{R})$ -transformation and define, therefore a non-degenerated symplectic form on the coset space  $LSL(2, \mathbb{R})/SL(2, \mathbb{R})$ . A global parametrization of this space is given by the metric degrees of freedom, i.e. by the currents  $j$ :

$$h = j_{(+)}(x^+) - 2j_{(0)}(x^+) \cdot x^- + j_{(-)}(x^+) \cdot (x^-)^2. \tag{4.23}$$

where

$$\begin{aligned} j_{(+)} &= B'D - BD' \\ j_{(-)} &= A'C - AC' \\ j_{(0)} &= -1/2(B'C + A'D - BC' - AD') \end{aligned} \tag{4.24}$$

This unravels the proper group/geometrical meaning of the 'hidden' symmetries as ordinary phase space symmetries, in contrast with the gauge-type interpretation arising in the light-cone formulation (described by the Virasoro group).

### 5. Chern-Simons gauge theory and WZW-models

As in the Virasoro case the integral of the one-form  $\Theta$  of the Kac-Moody group along 'trajectories'  $da = \dot{a} d\tau$  defines a (multivalued) action functional:

$$A \equiv \int \Theta = -\frac{k}{8\pi} \int d\tau \int_{S^1} d\sigma \langle a^{-1} \dot{a}', a^{-1} \dot{a} \rangle + 2\pi k \int d\tau \int_{S^1} \langle a^{-1} \dot{a}', h(\text{ad ln } a) a^{-1} \dot{a} \rangle. \tag{5.1}$$

The field model constructed this way will be called the Kac-Moody field model. From (5.1) it is definitely a two-dimensional field model. However, to get a more transparent expression for the action (5.1), it will be useful to rewrite it as a three-dimensional integral. Using the relation (3.14) and the Stokes theorem we obtain

$$\begin{aligned} A &= -\frac{k}{8\pi} \int d\tau \int_D \varepsilon^{ij} \partial_j \langle a^{-1} \partial_i a, a^{-1} \dot{a} \rangle + \frac{k}{8\pi} \int d\tau \int_D \varepsilon^{ij} \langle [a^{-1} \partial_i a, a^{-1} \partial_j a], a^{-1} \dot{a} \rangle \\ &= \frac{k}{8\pi} \int d\tau \int \varepsilon^{ij} \left\langle \partial_i a a^{-1}, \frac{d}{dt} (\partial_j a a^{-1}) \right\rangle. \end{aligned} \tag{5.2}$$

We immediately recognize the action (5.2) as that of the Chern-Simons theory [11] restricted to  $F_{ij} \equiv \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0$  on the disc ( $A_i \equiv \partial_i a a^{-1}$ ):

$$A = S_{CS}|_{F=0} = \frac{k}{8\pi} \int d\tau \int_D \varepsilon^{ij} A_i A_j. \tag{5.3}$$

The Kac-Moody field model constructed from the group  $\widetilde{LG}$  (the Lie algebra of which is the current algebra) is then equivalent to the Chern-Simons gauge theory on the disc. We have to stress that, in contrast with the first example, the Kac-Moody field model has been directly interpreted as a physical theory in the space-like formulation.

We can now wonder whether or not the Kac-Moody field model can also be related to some conformal model in the light-cone formulation. As first pointed out in [11] the CS theory on the disc is intimately related to WZW-models. We shall now recover this result in a simple way to show that the ‘hidden’ left-symmetries of the Kac-Moody field model correspond indeed to the well-known chiral left  $G$ -symmetry of WZW-models. The two-dimensional and multivalued Kac-Moody action functional (5.1) turns out to be the WZW-action [21] whenever the coordinates  $\sigma$  and  $\tau$  are interpreted as the light-cone coordinates  $x^-$  and  $x^+$ :

$$A = -\frac{k}{8\pi} \int d^2x \langle a^{-1} \partial_- a, a^{-1} \partial_+ a \rangle + 2\pi k \int d^2x \langle a^{-1} \partial_- a, h(\text{ad ln } a) a^{-1} \partial_+ a \rangle. \tag{5.4}$$

The covariant form of the expression (5.4) is the well-known WZW-action (had we chosen the interpretation of  $\sigma$  and  $\tau$  as  $x^+$  and  $x^-$  we would have not obtained the relative sign between the two terms of (5.2). This is, however, a matter of convention):

$$A = -\frac{k}{16\pi} \int d^2x \langle a^{-1} \partial_\mu a, a^{-1} \partial^\mu a \rangle + \frac{k}{24\pi} \int d^3y \epsilon^{ijk} \langle a^{-1} \partial_i a, a^{-1} \partial_j a a^{-1} \partial_k a \rangle. \tag{5.5}$$

For the sake of completeness we shall perform in WZW-models the calculation analogous to that of hidden symmetry in 2D-gravity. In so doing we shall be led to the well-known left  $G$ -current algebra [21, 22]. By construction, the action (5.5) possesses the right Kac-Moody symmetry generated by the right-invariant vector fields  $\tilde{X}_{a(x)}^R$  having  $j_{(i)}^- \equiv {}^i \tilde{X}_{a^i(x)\Theta}^R = k/2\pi \cdot \langle X_{a^i}, \partial^- a a^{-1} \rangle$  as Noether currents (3.24). The ‘hidden’ left-symmetry is generated by the characteristic subalgebra  $\mathcal{G}_\Theta$ , which is now given by

$$\mathcal{G}_\Theta = \langle \tilde{X}_{a^i}^L, i = 1, \dots, \dim G \rangle \tag{5.6}$$

where  $\tilde{X}_{a^i}^L$  are the zero modes of the vector fields  $\tilde{X}_{a^i(x)}^L$ . The conserved left-currents are then  $j_{(i)}^+ \sim \langle X_{a^i}, a^{-1} \partial^+ a \rangle$  ( $\partial^- j^+ = 0$ ).

### 5.1. Chern-Simons theory with sources

As is well known, the infinitesimal version of the two-cocycle (3.13) on the loop group  $LG$  gives rise to the centrally extended affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$ , which in the Cartan-Weyl basis takes the form (see for instance [23])

$$[H_n^i, H_m^j] = kn \delta^{ij} \delta_{n-m} \tag{5.7a}$$

$$[H_n^i, E_m^\alpha] = \alpha^i E_{n+m}^\alpha \tag{5.7b}$$

$$[E_n^\alpha, E_m^{-\alpha}] = \sum_i \frac{2(\alpha, \alpha^i)}{(\alpha, \alpha)} H_{n+m}^i + \frac{2k}{(\alpha, \alpha)} n \delta_{n-m} \tag{5.7c}$$

$$[E_n^\alpha, E_m^\beta] = \varepsilon(\alpha, \beta) E_{n+m}^{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root} \tag{5.7d}$$

where  $i, j = 1, \dots, r \equiv \text{rank of } g$ ,  $\alpha^i$  are the system of simple roots of  $g$  and  $\varepsilon(\alpha, \beta)$  are some integers. In the same way as the addition of a (trivial) Lie algebra two-cocycle,  $(-c'/12)n\delta_{n-m}$ , to the Virasoro algebra (corresponding to a redefinition of the  $L_0$  generator;  $c'/12$  must be an integer if  $L_0$  is a compact generator, although continuous

values of  $c'$  are allowed if we consider the universal covering group of  $\text{diff } S^1$ )

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m}$$

led to a non-trivial Lagrangian term, we can now consider trivial two-cocycles generated by the diagonal operators  $H_0^i (H_0^i \rightarrow H_0^i + \mu^i)$ . The Lie algebra commutators (5.7a, b, d) do not change but (5.7c) becomes

$$[E_n^\alpha, E_m^{-\alpha}] = \sum \frac{2(\alpha, \alpha^i)}{(\alpha, \alpha)} (H_{n+m}^i + \mu^i \delta_{n,-m}) + \frac{2k}{(\alpha, \alpha)} n \delta_{n,-m}. \tag{5.8}$$

Since the compact generators  $2(\alpha, \alpha^i)/(\alpha, \alpha) \cdot H_0^i$  have integer eigenvalues, the parameters  $\mu^i$  should satisfy the following integrability condition ( $\mu \equiv \sum_i \mu^i \alpha^i$ ):

$$\frac{2(\alpha, \mu)}{(\alpha, \alpha)} \equiv \langle \alpha, \mu \rangle \in \mathbb{Z} \tag{5.9}$$

otherwise the corresponding two-cocycles are not univalued. From now on the parameter  $\mu$  will be called *the weight of the extension*.

The contribution to the one-form  $\Theta$  of the corresponding two-cocycles on the loop group is

$$\begin{aligned} \Theta = \frac{d\zeta}{i\zeta} + \int d\sigma \left\langle \sum_i \mu^i H^i, a^{-1} da \right\rangle + 2\pi k \left[ \frac{-1}{16\pi^2} \int d\sigma \langle a^{-1} a', a^{-1} da \rangle \right. \\ \left. + \int d\sigma \langle a^{-1} a', h(\text{ad } \ln a) a^{-1} da \rangle \right] \end{aligned} \tag{5.10}$$

and the action functional is then of the form

$$A_{(k,\mu)} = \int \Theta = \frac{k}{8\pi} \int dt \int \varepsilon_D^{\mu} \left\langle \partial_t a a^{-1}, \frac{d}{dt} (\partial_t a a^{-1}) \right\rangle + \int dt \int_S d\sigma \left\langle \sum_i \mu^i H^i, a^{-1} \dot{a} \right\rangle. \tag{5.11}$$

We find this way a group theoretical description of the Chern-Simons theory in the presence of a source (compare, for instance, with [11]).

The action (5.11) does still have the right  $G$ -Kac-Moody symmetry, although the 'hidden' left-symmetry is no longer a  $G$  current algebra due to the presence of the additional term coming from the trival two-cocycle. In fact, the characteristic sub-algebra  $\mathcal{G}_\Theta$  or, what is the same, the kernel of the two-cocycle in (5.8) depends on the particular values of the parameters  $\mu^i$ . A simple inspection to the commutator (5.8) reveals that

$$\mathcal{G}_\Theta = \langle H_0^i, E_0^{\pm\alpha}; \forall i = 1, \dots, \text{rank } G, \forall \alpha / \langle \alpha, \mu \rangle = 0 \rangle. \tag{5.12}$$

If  $\mu = 0$ —the standard CS action— $\mathcal{G}_\Theta$  generates the ordinary left symmetry  $G$ . In the general case  $\mu = \sum m^i \lambda_i$ , where  $\lambda_i$  are the fundamental weights and  $m^i$  are integers (we shall restrict ourselves to  $m^i = 0, 1, 2, \dots$  because only these values will be allowed in the quantization),  $\mathcal{G}_\Theta$  generates some real parabolic subgroup  $P_G$  of  $G$ . If each  $m^i, i = 1, \dots, \text{rank } G$ , is different from zero,  $P_G$  is the maximal torus  $T = U(1) \otimes \dots \otimes U(1)$ . If  $\mu$  is just one of the fundamental weights  $\lambda_i$ ,  $P_G$  is:  $U(1) \otimes A_{i-1} \otimes A_{i-1}$  for  $A_i$ ;  $U(1) \otimes A_{i-1} \otimes B_{i-1}$  for  $B_i$ ;  $U(1) \otimes A_{i-1} \otimes C_{i-1}$  for  $C_i$ ;  $U(1) \otimes A_{i-1} \otimes D_{i-1}$  for  $D_i$ ; for  $E_6$ ,  $P_G$  is  $U(1) \otimes D_4$  if  $i = 1$  or  $6$ ,  $A_5$  if  $i = 2$ ,  $U(1) \otimes A_1 \otimes A_4$  if  $i = 3$  or  $5$ ,  $U(1) \otimes A_2 \otimes A_1 \otimes A_2$  if  $i = 4$ , etc (we have used the notation of [24]). For any degenerated weight— $m^j = 0$  for some  $j = 1, \dots, \text{rank } G$ —the determination of  $P_G$  comes also easily from the Dynkin diagrams.

5.2. Group theoretical quantization of the Chern–Simons theory with sources

We have seen in previous sections that our group quantization approach allows us to establish, at the classical level, a physical interpretation of centrally extended Lie groups as Lagrangian theories whose actions are given by the Maurer–Cartan form associated with the central generator. It is our aim in this subsection to quantize the theory using the new scheme. We now proceed to quantize the theory from the general prescription presented in section 3.

The phase space of a field model constructed from a Lie group  $\tilde{G}$  is given by  $G/\mathcal{G}_\Theta$ . In the present case, the corresponding phase spaces (also defined by the initial data surface  $t \equiv \tau = 0$ ) are then the coset (symplectic) spaces  $LG/P_G$  which, in turn, coincide with the co-adjoint orbits of the Kac–Moody group [12, 25]. Moreover, the action (5.11),  $A_{(k,\mu)}$ , is already in Hamiltonian form and the Poisson bracket algebra of the Kac–Moody Noether invariants  $j_n^-$ , which now close the algebra (5.7a, b, c; 5.8), is then associated with the action (5.11) much in the same way the Heisenberg–Weyl algebra

$$\{p, q\} = 1 \tag{5.13}$$

is associated with the ‘action’

$$\int p\dot{q} dt. \tag{5.14}$$

Following this analogy, the Hilbert space of the Chern–Simons models (5.11), is given by one (unitary) irreducible representation of the Kac–Moody algebra (5.7, 5.8) in the same way as the Hilbert space of the ‘action’ (5.14),  $L^2(\mathbb{R}, dx)$ , is given by the (unitary) irreducible representation of the algebra (5.13). In fact, and following again the general scheme presented in section 3, an explicit construction of the Hilbert space  $\mathcal{H}_{(k,\mu)}$  can be given in terms of the complex functions on  $LG$ , verifying the  $U(1)$ -equivariance condition  $\Xi\Psi = i\Psi$  as well as the polarization equations  $\tilde{X}^L\Psi = 0$ ,  $\tilde{X}^L \in \mathcal{P}$ , where the polarization  $\mathcal{P}$  is given by

$$\mathcal{P} = \langle \tilde{X}_{E_{n<0}}^L, \tilde{X}_{H_{n<0}}^L, \mathcal{G}_\Theta, \forall \alpha, \forall i = 1, \dots, \text{rank } G \rangle. \tag{5.15}$$

The condition for having a non-trivial space of polarized functions reads as follows:

$$0 \leq (\vartheta \cdot \mu) \leq k \tag{5.16}$$

where  $\vartheta$  is the highest root of the Lie algebra  $\mathfrak{g}$  of  $G$ . We can establish the condition above by realizing that for every trivially extended  $SU(2) \otimes U(1)$  subgroups of  $LG$  generated by

$$\langle \tilde{X}_{E_{-n}}^L, \tilde{X}_{E_n}^L, \tilde{X}_{H_0}^L, \Xi \rangle \quad n < 0$$

the space of polarized functions (with the restriction of  $\mathcal{P}$  to this  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  algebra) is globally defined only for (the winding number  $2(\alpha, \mu)/(\alpha, \alpha) + 2nk/(\alpha, \alpha)$  plays the role of the spin  $2j$ )

$$(\alpha \cdot \mu) + nk \geq 0 \quad \forall n \geq 0 \quad \forall \alpha. \tag{5.17a}$$

If  $n > 0$ , the restriction of  $\mathcal{P}$  to the  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  subalgebra is now different and the condition for having a non-empty space of polarized functions is

$$(\alpha \cdot \mu) + nk \leq 0 \quad \forall n < 0 \quad \forall \alpha. \tag{5.17b}$$

Both conditions are equivalent to (5.16).

Among the polarized functions there is one (a highest-weight vector  $|0\rangle$ ) that is annihilated by the operators

$$\tilde{X}_{E_{n>0}}^L \quad \tilde{X}_{H_{n>0}}^L \quad \tilde{X}_{\mathfrak{g}_0}^L.$$

In particular,  $|0\rangle$  is annihilated by the operators  $\tilde{X}_{H_0^i}^R$ . Now, redefining the operators  $\tilde{X}_{H_0^i}^R - \mu^i \Xi \equiv \tilde{X}_{H_0^i}^R$ , to re-establish the standard commutators (5.7c) of the Kac-Moody algebras, the primed generators verify

$$\tilde{X}_{H_0^i}^R |0\rangle = \mu^i |0\rangle. \tag{5.18}$$

The expression above shows that the parameters  $\mu^i$  of the trivial extension that led to the insertion of a source for the CS theory are the weights of the corresponding quantum representation. Thus, the Hilbert space  $\mathcal{H}_{(k,\mu)}$  is given by an integrable highest-weight representation of level  $2k/\vartheta^2$  (or  $k$  if we choose  $\vartheta^2=2$ ) and weight  $\mu$ . Since the vacuum is also annihilated by the operators  $\tilde{X}_{\mathfrak{g}_0}^R$ , the isotropy group of the vacuum  $|0\rangle \equiv |\mu\rangle$  is then given by the parabolic subgroup  $P_G$  associated with the weight  $\mu$ . This result provides an almost one-to-one correspondence between the classical actions  $A_{(k,\mu)}$ , the phase space  $LG/P_G$  and the corresponding Hilbert spaces  $\mathcal{H}_{(k,\mu)}$ , and generalize, in some sense, the Borel-Weil theorem [12], which only consider the phase space  $LG/T$ , where  $T$  is the maximal torus.

We now briefly comment on the quantization of the  $SL(2, \mathbb{R})$ -Kac-Moody field model (i.e. the  $SL(2, \mathbb{R})$ -Chern-Simons theory, or the gauge-fixed Polyakov action, in the space-like formulation). If we try to extend the discussion of the theories with compact gauge group to the  $SL(2, \mathbb{R})$ -field model we face the problem that the condition for having a unitary, standard highest-weight representation implies a vanishing central charge (see [26])\* . We arrive this way at the apparently disappointing conclusion that there is not a consistent quantum description of the model. However, from the physical point of view and, in the context of the 2D-induced gravity interpretation, this is not an unavoidable drawback. Since the starting point has been a gauge-fixed action, the physical Hilbert space should be then defined as a constrained subspace of the carrier space of the irreducible representation. The corresponding constraint is:  $\hat{j}_{(-)} = \Lambda/2$ , i.e. the constant curvature condition ( $R \equiv 4\partial^2 h = \Lambda/2$ ) coming from the covariant Lagrangian  $\mathcal{L} = c/96\pi \sqrt{g}(R\Delta^{-1}R + \Lambda)$ . The constraint leads then to the standard quantum Hamiltonian reduction [28] allowing reinterpretation of the (constrained) theory, in the space-like formulation, in terms of the Virasoro group.

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\* Remarkably, it is still possible to construct a new class of unitary, irreducible representations of the  $SL(2, \mathbb{R})$ -Kac-Moody group with zero central charge [26]. A physical interpretation of these (exceptional) representations, along the lines of the present work, has been given in terms of the quantization of an ultralocal field model for gravity [27].

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